(4)

A NEW METHOD OF INVESTIGATING THE SUMS OF INFINITE SERIES.

By the Rev. Mr. VINCE, A. M. of CAMBRIDGE.

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A NEW METHOD, &c.

REV. MR. MATY, SEC. R. S.

SIR,

HAVING lately diſcovered ſome very eaſy methods of inveſtigating the ſums of certain infinite ſeries, I have taken the liberty of requeſting the favour of you to preſent them to the Royal Society. I have divided the ſubject into three parts: the firſt contains a new and general method of finding the ſum of thoſe ſeries which DE MOIVRE has found in one or two particular caſes; but whoſe method, although it be in appearance general, will, upon trial, be found to be abſolutely impracticable. The ſecond contains the ſummation of certain ſeries, the laſt differences of whoſe numerators become equal to nothing. The third contains obſervations on a *correction* which is neceſſary in inveſtigating the ſums of certain ſeries by collecting two terms into one, with its application to a variety of caſes.

I am, &c.

Cambridge,
May 3, 1782.

PART I.

LEM. I.

Let r be any whole number, and then the fluent of $\frac{\dot{x}}{1+x^r}$ *can always be exhibited by circular arcs and logarithms; but when* $x=1$, *the fluent of the ſame fluxion will be expreſſed by the infinite ſeries* $1-\frac{1}{r+1}+\frac{1}{2r+1}-\frac{1}{3r+1}+$ &c. *the ſum of this ſeries therefore can always be found by circular arcs and logarithms.*

LEM. II.

To find the ſum of the infinite ſeries $\frac{a}{1\cdot\overline{r+1}}-\frac{a+b}{\overline{r+1}\cdot\overline{2r+1}}+\frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}}-$ &c.

Aſſume $1-\frac{1}{r+1}+\frac{1}{2r+1}-\frac{1}{3r+1}+$ &c. . $=S$; therefore,

(A) $1-\frac{1}{1\cdot\overline{r+1}}+\frac{r+1}{\overline{r+1}\cdot\overline{2r+1}}-\frac{2r+1}{\overline{2r+1}\cdot\overline{3r+1}}+$ &c. . . . $=S$.

In the firſt ſeries, add together the 1ſt and 2d, the 2d and 3d, &c. &c. terms, and the reſulting ſeries will evidently be equal to twice that ſeries *minus* the firſt term; therefore,

(B), $\frac{r}{1\cdot\overline{r+1}}-\frac{r}{\overline{r+1}\cdot\overline{2r+1}}+\frac{r}{\overline{2r+1}\cdot\overline{3r+1}}-$ &c. . . . $=2S-1$.

Now $\left(\frac{A}{r}\right)\frac{1}{r}-\frac{\frac{1}{r}}{1\cdot\overline{r+1}}+\frac{1+\frac{1}{r}}{\overline{r+1}\cdot\overline{2r+1}}-\frac{2+\frac{1}{r}}{\overline{2r+1}\cdot\overline{3r+1}}+$ &c. . . . $=\frac{S}{r}$,

$$\text{or}\ \left.\begin{array}{l}\frac{1}{\overline{r+1}\cdot\overline{2r+1}}-\frac{2}{\overline{2r+1}\cdot\overline{3r+1}}+\&c.\ldots\\+\frac{1}{r}-\frac{\frac{1}{r}}{1\cdot\overline{r+1}}+\frac{\frac{1}{r}}{\overline{r+1}\cdot\overline{2r+1}}-\&c.\ldots\end{array}\right\}=\frac{S}{r};$$

Now the ſum of the lower ſeries, omitting the firſt term, is equal to $-B$ divided by r^2, or $= -\frac{2S-1}{r^2}$; hence, by tranſpoſition, and, multiplying both ſides by b, we ſhall have,

$$\frac{b}{\overline{r+1}\cdot\overline{2r+1}} - \frac{2b}{\overline{2r+1}\cdot\overline{3r+1}} + \&\text{c.}\ldots = \frac{bS}{r} + \frac{2bS-\overline{r+1}\cdot b}{r^2};$$

alſo by multiplying B by $\frac{a}{r}$ we have

$$\frac{a}{1\cdot\overline{r+1}} - \frac{a}{\overline{r+1}\cdot\overline{2r+1}} + \frac{a}{\overline{2r+1}\cdot\overline{3r+1}} - \&\text{c.}\ldots = \frac{2aS-a}{r};$$

ſubtract the laſt equation but one from the laſt, and we ſhall have

$$\frac{a}{1\cdot\overline{r+1}} - \frac{a+b}{\overline{r+1}\cdot\overline{2r+1}} + \frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}} - \&\text{c.}\ldots = \frac{\overline{2ra-\overline{r+2}\cdot b}\times S-ra+\overline{r+1}\cdot b}{r^2}.$$

Cor. 1. Hence it appears, that the ſum of this ſeries can never be exhibited in finite terms, except $a:b$ as $r+2:2r$, in which caſe the ſum is equal to $\frac{a}{r+2}$.

Hence, if $a=3$, $b=2$, then $r=1$; $\therefore \frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \&\text{c.}\ldots = 1$;

if $a=1$, $b=4$, then $r=\frac{4}{3}$; $\therefore \frac{5}{3\cdot 7} - \frac{9}{7\cdot 11} + \frac{13}{11\cdot 15} - \frac{17}{15\cdot 19} + \&\text{c.}\ldots = \frac{1}{6}$;

if $a=4$, $b=3$, then $r=\frac{6}{5}$; $\therefore \frac{4}{5\cdot 11} - \frac{7}{11\cdot 17} + \frac{10}{17\cdot 23} - \frac{13}{23\cdot 29} + \&\text{c.}\ldots = \frac{1}{20}$;

Cor 2. Put $a=c-b$, and we ſhall have, after tranſpoſition,

$$\frac{c}{\overline{r+1}\cdot\overline{2r+1}} - \frac{c+b}{\overline{2r+1}\cdot\overline{3r+1}} + \&\text{c.}\ldots = \frac{\overline{\overline{3r+2}\cdot b-2rc}\times S-\overline{2r+1}\ \ b+rc}{r^2} + \frac{c-b}{r+1}.$$

P R O P. I.

To find the ſum of the infinite ſeries $\frac{m}{1\cdot\overline{r+1}\cdot\overline{2r+1}} + \frac{m+n}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}} - \frac{m+2n}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}} + \&\text{c.}$

Every ſeries of this kind may be reſolved into the following ſeries $\frac{a}{1.\overline{r+1}} - \frac{a+b}{\overline{r+1}.\overline{2r+1}} + \frac{a+2b}{\overline{2r+1}.\overline{3r+1}} - \frac{a+3b}{\overline{3r+1}.\overline{4r+1}} +$ &c. for if we reduce two terms of this ſeries into one, it will become

$$\frac{2ar-b}{1.\overline{r+1}.\overline{2r+1}} + \frac{2ra+\overline{2r-1}.b}{\overline{2r+1}.\overline{3r+1}.\overline{4r+1}} + \frac{2ra+\overline{4r-1}.b}{\overline{4r+1}.\overline{5r+1}.\overline{6r+1}} + \text{\&c.}$$

where the denominators being the ſame as in the given ſeries, and the numerators alſo in arithmetic progreſſion, we have only to take a and b ſuch quantities that the reſpective numerators may be alſo equal; aſſume, therefore, $2ra-b=m$, $2ra+\overline{2r-1}.b=m+n$; therefore, $b=\frac{n}{2r}$, $a=\frac{2rm+n}{4r^2}$, which ſubſtituted for a and b in LEM. 2. gives

$$\frac{m}{1.\overline{r+1}.\overline{2r+1}} + \frac{m+n}{\overline{2r+1}.\overline{3r+1}.\overline{4r+1}} + \frac{m+2n}{\overline{4r+1}.\overline{5r+1}.\overline{6r+1}} + \text{\&c.}\ldots$$

$$= \frac{2rm-\overline{r+1}.n}{2r^3} \times S + \frac{\overline{2r+1}.n-2rm}{4r^3}.$$

Let $r=1$, and we have

$$\frac{m}{1.2.3} + \frac{m+n}{3.4.5} + \frac{m+2n}{5.6.7} + \text{\&c.}\ldots = \overline{m-n}.S + \frac{3n-2m}{4}.$$

If $m=1$, $n=3$, $\frac{1}{1.2.3} + \frac{4}{3.4.5} + \frac{7}{5.6.7} + \text{\&c.}\ldots = \frac{7}{4} - 2S$;

$m=1$, $n=0$, $\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \text{\&c.}\ldots = S - \frac{1}{2}$.

Let $r=2$, and the ſeries becomes

$$\frac{m}{1.3.5} + \frac{m+n}{5.7.9} + \frac{m+2n}{9.11.13} + \text{\&c.}\ldots = \frac{4m-3n}{16} \times S + \frac{5n-4m}{32}.$$

If $m=1$, $n=1$, $\frac{1}{1.3.5} + \frac{2}{5.7.9} + \frac{3}{9.11.13} + \text{\&c.}\ldots = \frac{S}{16} + \frac{1}{32}$;

$m=1$, $n=0$, $\frac{1}{1.3.5} + \frac{1}{5.7.9} + \frac{1}{9.11.13} + \text{\&c.}\ldots = \frac{S}{4} - \frac{1}{8}$.

Let $r=5$, and we ſhall have

$$\frac{m}{1.6.11} + \frac{m+n}{11.16.21} + \frac{m+2n}{21.26.31} + \text{\&c.}\ldots = \frac{5m-3n}{125} \times S + \frac{11n-10m}{500}.$$

If $m=1,\ n=1,\ \frac{1}{1.6.11}+\frac{2}{11.16.21}+\frac{3}{21.26.31}+\&c.\ldots=\frac{2}{125}\times S+\frac{1}{500}$;

$m=1,\ n=0,\ \frac{1}{1.6.11}+\frac{1}{11.16.21}+\frac{1}{21.26.31}+\&c.\ldots=\frac{S}{25}-\frac{1}{50}$.

Cor. If $2r : r+1 :: n : m$, the ſum of the ſeries can be accurately found, and will be equal to $\frac{m}{2r.\overline{r+1}}$. Let therefore $m=r+1$, and then $n=2r$, conſequently

$$\frac{1}{1.\overline{2r+1}}+\frac{1}{\overline{2r+1}.\overline{4r+1}}+\frac{1}{\overline{4r+1}.\overline{6r+1}}+\&c.\ldots=\frac{1}{2r};$$

which is alſo known from other principles.

P R O P. II.

To find the ſum of the infinite ſeries $\frac{m}{\overline{r+1}.\overline{2r+1}.\overline{3r+1}}+\frac{m+n}{\overline{3r+1}.\overline{4r+1}.\overline{5r+1}}+\frac{m+2n}{\overline{5r+1}.\overline{6r+1}.\overline{7r+1}}+\&c.$

This ſeries reſolves itſelf into

$$\frac{c}{\overline{r+1}.\overline{2r+1}}-\frac{c+b}{\overline{2r+1}.\overline{3r+1}}+\frac{c+2b}{\overline{3r+1}.\overline{4r+1}}-\&c.;$$

for by reduction, as before, it becomes

$$\frac{2cr-\overline{r+1}.b}{\overline{r+1}.\overline{2r+1}.\overline{3r+1}}+\frac{2cr+\overline{r-1}.b}{\overline{3r+1}.\overline{4r+1}.\overline{5r+1}}+\frac{2cr+\overline{3r-1}.b}{\overline{5r+1}.\overline{6r+1}.\overline{7r+1}}+\&c.$$

where the denominators are the ſame as in the given ſeries, and the numerators in arithmetic progreſſion; aſſume therefore $2cr-\overline{r+1}.b=m$, $2cr+\overline{r-1}.b=m+n$, hence $b=\frac{n}{2r}$, $c=\frac{2rm+\overline{r+1}.n}{4r^2}$, which, ſubſtituted in cor. 2. LEM. 2. give

$$\frac{m}{\overline{r+1}.\overline{2r+1}.\overline{3r+1}}+\frac{m+n}{\overline{3r+1}.\overline{4r+1}.\overline{5r+1}}+\frac{m+3n}{\overline{5r+1}.\overline{6r+1}.\overline{7r+1}}+\&c.\ldots$$
$$=\frac{\overline{2r+1}.n-2rm}{2r^3}\times S+\frac{2rm-\overline{3r+1}.n}{4r^3}+\frac{2rm-\overline{r-1}.n}{4r^2.\overline{r+1}}.$$

Cor. 1. In prop. 1. ſubſtitute *a* for *m*, and 2*b* for *n*, and we have

$$\frac{a}{1\cdot\overline{r+1}\cdot\overline{2r+1}}+\frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{a+4b}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\&c.\ldots=$$
$$\frac{ra-\overline{r+1}\cdot b}{r^3}\times S+\frac{\overline{2r+1}\cdot b-ra}{2r^3}.$$

Alſo in this prop. ſubſtitute $a+b$ for *m*, and 2*b* for *n*, and we have

$$\frac{a+b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+\frac{a+3b}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}+\&c.\ldots=$$
$$\frac{\overline{r+1}\cdot b-ra}{r^3}\times S+\frac{ra-\overline{2r+1}\cdot b}{2r^3}+\frac{ra+b}{2r^2\times\overline{r+1}}.$$

Subtract this latter ſeries from the former, and

$$\frac{a}{1\cdot\overline{r+1}\cdot\overline{2r+1}}-\frac{a+b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+\frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}-\&c.\ldots=$$
$$\frac{2ra-\overline{r+1}\cdot 2b}{r^3}\times S+\frac{\overline{2r+1}\cdot b-ra}{r^3}-\frac{ra+2b}{2r^2\times\overline{r+1}}.$$

Let $r=1$, and we have

$$\frac{a}{1\cdot2\cdot3}-\frac{a+b}{2\cdot3\cdot4}+\frac{a+2b}{3\cdot4\cdot5}-\&c.\ldots=\overline{2a-4b}\times S+\frac{11b-5a}{4}.$$

If $a=1$, $b=0$, $\frac{1}{1\cdot2\cdot3}-\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}-\&c.\ldots=2S-\frac{5}{4}$;

$a=1$, $b=2$, $\frac{1}{1\cdot2\cdot3}-\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}-\&c.\ldots=\frac{17}{4}-6S$.

Let $r=3$, and we have

$$\frac{a}{1\cdot4\cdot7}-\frac{a+b}{4\cdot7\cdot10}+\frac{a+2b}{7\cdot10\cdot13}-\&c.\ldots=\frac{6a-8b}{27}\times S+\frac{53b-33a}{216}.$$

If $a=1$, $b=0$, $\frac{1}{1\cdot4\cdot7}-\frac{1}{4\cdot7\cdot10}+\frac{1}{7\cdot10\cdot13}-\&c.\ldots=\frac{2}{9}S-\frac{11}{72}$;

$a=1$, $b=1$, $\frac{1}{1\cdot4\cdot7}-\frac{2}{4\cdot7\cdot10}+\frac{3}{7\cdot10\cdot13}-\&c.\ldots=\frac{5}{54}-\frac{2}{27}S$.

If, inſtead of ſubſtituting in prop. 1. 2*b* and *a* for *n* and *m*, we had ſubſtituted two other quantities, as 2*r* and *s*, and then proceeded as above, a ſeries would have been formed, the numerators of whoſe alternate terms would have formed each a a ſeparate arithmetic progreſſion.

If the latter ſeries had been *added* to the former a ſeries would have been formed whoſe terms would have been all poſitive; but as I purpoſe, in the ſecond part of this paper, to give a general method of ſumming all ſuch ſeries, I ſhall not ſtop here to apply this method of inveſtigation.

Cor. 2. In propoſition 2. ſubſtitute a for m, and $2b$ for n, and we ſhall have

$$\frac{a}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+\frac{a+2b}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}+\&c.\ldots=$$

$$\frac{\overline{2r+1}\cdot b-ra}{r^3}\times S+\frac{ra-\overline{3r+1}\cdot b}{2r^3}+\frac{ra-\overline{r-1}\cdot b}{2r^2\cdot\overline{r+1}}.$$

Alſo in prop. 1. write $a-b$ for m, and $2b$ for n, and there reſults

$$\frac{a+b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{a+3b}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\&c.\ldots=$$

$$\frac{ra-\overline{2r+1}\cdot b}{r^3}\times S+\frac{\overline{r+1}\cdot b-ra}{2r^3}-\frac{a-b}{\overline{r+1}\cdot\overline{2r+1}}.$$

Subtract this latter ſeries from the former, and we ſhall have

$$\frac{a}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}-\frac{a+b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{a+2b}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}-\&c.\,.$$

$$=\frac{\overline{2r+1}\cdot 2b-2ra}{r^3}\times S+\frac{ra-\overline{3r+1}\cdot b}{r^3}+\frac{ra-\quad\overline{-1}\cdot b}{2r^2\cdot\overline{r+1}}+\frac{a-b}{\overline{r+1}\cdot\overline{2r+1}}.$$

P R O P. III.

To find the ſum of the infinite ſeries $\frac{m}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+$ $\frac{m+n}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}+\frac{m+2n}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}}+$ &c.

This ſeries reſolves itſelf into

$$\frac{a}{1\cdot\overline{r+1}\cdot\overline{2r+1}}-\frac{a+b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+\frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}-\&c.$$

for by reduction it becomes

$$\frac{3ra-b}{1\,.\,\overline{r+1}\,.\,\overline{2r+1}\,.\,\overline{3r+1}}+\frac{3ar+\overline{4r-1}\,.\,b}{\overline{2r+1}\,.\,\overline{3r+1}\,.\,\overline{4r+1}\,.\,\overline{5r+1}}+$$

$$\frac{3ar+\overline{8r-1}\,.\,b}{\overline{4r+1}\,.\,\overline{5r+1}\,.\,\overline{6r+1}\,.\,\overline{7r+1}}+\&c.$$

where the numerators are in arithmetic progreſſion, and the denominators the ſame as in the given ſeries; aſſume therefore $3ra-b=m$, $3ra+\overline{4r-1}\,.\,b=m+n$, hence $b=\frac{n}{4r}$, $a=\frac{4rm+n}{12r^2}$; ſubſtitute theſe values into cor. 1. prop. 2. and we have

$$\frac{m}{1\,.\,\overline{r+1}\,.\,\overline{2r+1}\,.\,\overline{3r+1}}+\frac{m+n}{\overline{2r+1}\,.\,\overline{3r+1}\,.\,\overline{4r+1}\,.\,\overline{5r+1}}+$$

$$\frac{m+2n}{\overline{4r+1}\,.\,\overline{5r+1}\,.\,\overline{6r+1}\,.\,\overline{7r+1}}+\&c.\ \ldots=\frac{4rm-\overline{3+2}\,.\,n}{6r^4}\times S+$$

$$\frac{\overline{3r+1}\,.\,n-2rm}{6r^4}-\frac{rm+n}{6r^3\,.\,\overline{r+1}}.$$

Let $r=1$, and we have

$$\frac{m}{1.2.3.4}+\frac{m+n}{3.4.5.6}+\frac{m+2n}{5.6.7.8}+\&c.\ldots=\frac{4m-5n}{6}\times S+\frac{7n-5m}{12}.$$

If $m=1, n=1$, $\frac{1}{1.2.3.4}+\frac{1}{3.4.5.6}+\frac{1}{5.6.7.8}+\&c.\ldots=\frac{2}{3}S-\frac{5}{12}$;

$m=1, n=1$, $\frac{1}{1.2.3.4}+\frac{2}{3.4.5.6}+\frac{3}{5.6.7.8}+\&c.\ldots=\frac{1}{6}-\frac{1}{6}S$;

$m=7, n=5$, $\frac{7}{1.2.3.4}+\frac{12}{3.4.5.6}+\frac{17}{5.6.7.8}+\&c.\ldots=\frac{1}{2}S$;

Let $r=2$, and we have

$$\frac{m}{1.3.5.7}+\frac{m+n}{5.7.9.11}+\frac{m2n}{9.11.13.15}+\&c.\ldots=\frac{m-n}{12}\times S-\frac{19n-16m}{288}.$$

If $m=1, n=3$, $\frac{1}{1.3.5.7}+\frac{4}{5.7.9.11}+\frac{7}{9.11.13.15}+\&c.\ldots=\frac{41}{288}-\frac{1}{6}S$;

$m=1, n=0$, $\frac{1}{1.3.5.7}+\frac{1}{5.7.9.11}+\frac{1}{9.11.13.15}+\&c.\ldots=\frac{1}{12}S-\frac{1}{18}$.

$m=19, n=16$. $\frac{19}{1.3.5.7}+\frac{35}{5.7.9.11}+\frac{51}{9.11.13.15}+\&c.\ldots=\frac{1}{4}S$.

Cor. If $n:m$ as $4r:3r+2$, the ſum of the ſeries can be accurately had; let therefore $n=4r$ and $m=3r+2$, and we ſhall have

$$\frac{3r+2}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+\frac{7r+2}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}+\&c.\ldots=\frac{1}{2r\cdot\overline{r+1}}.$$

If $r=1$, $\frac{5}{1\cdot2\cdot3\cdot4}+\frac{9}{3\cdot4\cdot5\cdot6}+\frac{13}{5\cdot6\cdot7\cdot9}+\&c.\ldots=\frac{1}{4}$*;

$r=2$, $\frac{1}{1\cdot3\cdot5\cdot7}+\frac{2}{5\cdot7\cdot9\cdot11}+\frac{3}{9\cdot11\cdot13\cdot15}+\&c.\ldots=\frac{1}{96}$

$r=6$, $\frac{5}{1\cdot7\cdot13\cdot19}+\frac{11}{13\cdot19\cdot25\cdot31}+\frac{17}{25\cdot31\cdot37\cdot43}+\&c.\ldots=\frac{1}{336}$.

P R O P. IV.

To find the ſum of the infinite ſeries $\frac{m}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{m+n}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\frac{m+2n}{\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}\cdot\overline{8r+1}}+\&c.$

This ſeries reſolves itſelf into

$$\frac{a}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}-\frac{a+b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{a+2b}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}-\&c.$$

for by reduction it becomes $\frac{3ra-\overline{r+1}\cdot b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{3ra+\overline{3r-1}\cdot b}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\frac{3ra+\overline{7r-1}\cdot b}{\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}\cdot\overline{8r+1}}+\&c.$ where the denominators are the ſame as in the given ſeries, and the numerators alſo in arithmetic progreſſion; put therefore $3ra-\overline{r+1}\cdot b=m$, $3ra+\overline{3r-1}\cdot b=m+n$, hence $b=\frac{n}{4r}$, $a=\frac{4rm+\overline{r+1}\cdot n}{12r^2}$, which, ſubſtituted in cor. 2. prop. 2. give

$$\frac{m}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{m+n}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\&c.\ldots=$$

$$\frac{\overline{5r+2}\cdot n-4rm}{6r^4}\times S+\frac{2rm-\overline{4r+1}\cdot n}{6r^4}+\frac{2rm-\overline{r-2}\cdot n}{12\cdot r^3\cdot\overline{r+1}}+\frac{4rm-\overline{2r-1}\cdot n}{12r^2\cdot\overline{r+1}\cdot\overline{2r+1}}.$$

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* Vide DE MOIVRE's Miſ. Anal. pag. 134.

Cor. 1. In prop. 3. write a for m and $2b$ for n, and we have

$$\frac{a}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+\frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}+\&c.\ldots=$$

$$\frac{2r\text{[illegible]}\overline{3r+2}\cdot b}{3r^4}\times S+\frac{\overline{3r+1}\cdot b-ra}{3r^4}-\frac{ra+2b}{6r^3\cdot\overline{r+1}}.$$

Alſo in this prop. write $a+b$ for m, and $2b$ for n, and we have

$$\frac{a+b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{a+3b}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\&c.\ldots=$$

$$\frac{\overline{3r+2}\cdot b-2ra}{3r^4}\times S+\frac{ra-\overline{3r+1}\cdot b}{3r^4}+\frac{ra+2b}{6r^3\cdot\overline{r+1}}+\frac{2ra+b}{6r^2\cdot\overline{r+1}\cdot\overline{2r+1}}.$$

ſubtract this latter ſeries from the former, and we have

$$\frac{a}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}-\frac{a+b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\&c.\ldots=$$

$$\frac{4ra-\overline{3r+2}\cdot 2b}{3r^4}\times S+\frac{\overline{3r+1}\cdot 2b-2ra}{3r^4}-\frac{ra+2b}{3r^3\cdot\overline{r+1}}-\frac{2ra+b}{6r^2\cdot\overline{r+1}\cdot\overline{2r+1}}.$$

Let $r=1$, and we have

$$\frac{a}{1\cdot2\cdot3\cdot4}-\frac{a+b}{2\cdot3\cdot4\cdot5}+\frac{a+2b}{3\cdot4\cdot5\cdot6}-\&c.\ldots=\frac{4a-10b}{3}\times S+\frac{83b-32a}{36}.$$

If $a=1$, $b=0$, $\frac{1}{1\cdot2\cdot3\cdot4}-\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}-\&c.\ldots=\frac{4}{3}S-\frac{8}{9}.$

$a=3$, $b=1$, $\frac{1}{1\cdot2\cdot4}-\frac{1}{2\cdot3\cdot5}+\frac{1}{3\cdot4\cdot6}-\&c.\ldots=\frac{2}{3}S-\frac{13}{36}.$

Let $r=3$, and we have

$$\frac{a}{1\cdot4\cdot7\cdot10}-\frac{a+b}{4\cdot7\cdot10\cdot13}+\frac{a+2b}{7\cdot10\cdot13\cdot16}-\&c.\ldots=\frac{12a-22b}{243}\times S+\frac{1027b-156a}{13\cdot81\cdot56}$$

If $a=1$, $b=1$, $\frac{1}{1\cdot4\cdot7\cdot10}-\frac{2}{4\cdot7\cdot10\cdot13}+\frac{3}{7\cdot10\cdot13\cdot16}-\&c.\ldots=\frac{871}{3\cdot81\cdot56}-\frac{10}{3\cdot81}S.$

$a=4$, $b=3$, $\frac{1}{1\cdot7\cdot10}-\frac{1}{4\cdot10\cdot13}+\frac{1}{7\cdot13\cdot16}-\&c.\ldots=\frac{13}{72}-\frac{2}{27}S.$

Cor. 2. If $a:b$ as $3r+2:2r$, the ſum of the ſeries can be accurately found; take $\therefore a=3r+2$, and $b=2r$, and we ſhall have

$$\frac{3r+2}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}-\frac{5r+2}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\&c.\ldots=\frac{1}{\overline{r+1}\cdot\overline{2r+1}}.$$

If $r=1$, $\frac{5}{1\cdot2\cdot3\cdot4}-\frac{7}{2\cdot3\cdot4\cdot5}+\frac{9}{3\cdot4\cdot5\cdot6}-\&c.\ldots=\frac{1}{60}$;

$r=2$, $\frac{2}{1\cdot3\cdot5\cdot7}-\frac{3}{3\cdot5\cdot7\cdot9}+\frac{4}{5\cdot7\cdot9\cdot11}-\&c.\ldots=\frac{1}{15}$;

$r=3$, $\frac{11}{1\cdot4\cdot7\cdot10}-\frac{17}{4\cdot7\cdot10\cdot13}+\frac{23}{7\cdot10\cdot13\cdot16}-\&c.\ldots=\frac{1}{28}$.

P R O P. V.

To find the ſum of the infinite ſeries $\frac{m}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{m+n}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\frac{m+2n}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}\cdot\overline{8r+1}}+\&c.$

This ſeries reſolves itſelf into $\frac{a}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}-\frac{a+b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}-\&c.$; for by reduction this ſeries becomes $\frac{4ra-b}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{4ra+\overline{br-1}\cdot b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\frac{4ra+\overline{12r-1}\cdot b}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}\cdot\overline{8r+1}}+\&c.$ where the numerators are in arithmetic progreſſion, and the denominators the ſame as in the given ſeries; aſſume therefore $4ra-b=m$, $4ra+\overline{6r-1}\cdot b=m+n$, hence $b=\frac{n}{6r}$, $a=\frac{6rm+n}{24r^2}$, which values being ſubſtituted in cor. 1. prop. 4. give

$$\frac{m}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{m+n}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\&c.\ldots=\frac{2rm-\overline{2r+1}\cdot n}{6r^5}\times S+\frac{\overline{4r+1}\cdot n-2rm}{12r^5}-\frac{6rm+9n}{72\cdot r^4\cdot\overline{r+1}}-\frac{2rm+n}{24r^3\cdot\overline{r+1}\cdot\overline{2r+1}}.$$

Let $r=1$, and we have

$$\frac{m}{1\cdot2\cdot3\cdot4\cdot5}+\frac{m+n}{3\cdot4\cdot5\cdot6\cdot7}+\frac{m+2n}{5\cdot6\cdot7\cdot8\cdot9}+\&c.\ldots=\frac{2m-3n}{6}\times S+\frac{25n-16m}{72}.$$

If $m=1, n=0, \frac{1}{1.2.3.4.5}+\frac{1}{3.4.5.6.7}+\frac{1}{5.6.7.8.9}+\&c.\ldots=\frac{1}{3}S-\frac{2}{9}$;

$m=1, n=1, \frac{1}{1.2.3.4.5}+\frac{2}{3.4.5.6.7}+\frac{3}{5.6.7.8.9}+\&c.\ldots=\frac{1}{8}-\frac{1}{6}S$;

$m=4, n=2, \frac{1}{1.2.3.5}+\frac{1}{3.4.5.7}+\frac{1}{5.6.7.9}+\&c.\ldots=\frac{1}{3}S-\frac{7}{36}$.

$m=25, n=16, \frac{25}{1.2.3.4.5}+\frac{41}{3.4.5.6.7}+\frac{57}{5.6.7.8.9}+\&c.\ldots=\frac{1}{3}S$.

Let $r=2$, and we have $\frac{m}{1.3.5.7.9}+\frac{m+n}{5.7.9.11.13}+\frac{m+2n}{9.11.13.15.17}+\&c.\ldots=\frac{4m-5n}{192}\times S+\frac{59n-44m}{24.120}$.

If $m=1, n=0, \frac{1}{1.3.5.7.9}+\frac{1}{5.7.9.11.13}+\frac{1}{9.11.13.15.17}+\&c.\ldots=\frac{1}{48}S-\frac{11}{24.30}$;

$m=3, n=2, \frac{1}{1.5.7.9}+\frac{1}{5.9.11.13}+\frac{1}{9.13.15.17}+\&c.\ldots=\frac{1}{96}S-\frac{7}{24.60}$.

Cor. If $n:m$ as $2r:2r+1$, the ſum of the ſeries can be accurately found; aſſume therefore $n=2r$, $m=2r+1$, and we have

$$\frac{1}{1.\overline{r+1}.\overline{3r+1}.\overline{4r+1}}+\frac{1}{\overline{2r+1}.\overline{3r+1}.\overline{5r+1}.\overline{6r+1}}+\&c.\ldots=\frac{1}{6.r.\overline{r+1}.\overline{2r+1}}.$$

If $r=1, \frac{1}{1.2.4.5}+\frac{1}{3.4.6.7}+\frac{1}{5.6.8.9}+\&c.\ldots=\frac{1}{36}$;

$r=3, \frac{1}{1.4.10.13}+\frac{1}{7.10.16.19}+\frac{1}{13.16.22.25}+\&c.\ldots\frac{1}{504}$.

Having thus far explained the method of ſummation of ſuch ſeries as I propoſed to treat of in the firſt part of this paper, I truſt it is not neceſſary to ſay any thing further, as the ſame method of proceeding will manifeſtly continue the ſeries to any propoſed number of factors in the denominator; I ſhall therefore conclude with pointing out a remarkable property of thoſe ſeries whoſe ſum can be accurately found: that when the number of factors in the denominator is *even*, the numerator is always equal to the ſum of the two middle factors; and when the number of factors be *odd*, the numerator will be equal to the middle factor, and conſequently will take it out of the denominator, and leave a ſeries whoſe numerators are unity, and whoſe denominators want the middle factor.

The method of ſummation of ſeries here made uſe of may alſo be applied in inveſtigating the ſums of a great variety of other ſeries; but as a further application of this method would carry us beyond the limits to which this paper muſt be confined, I ſhall re-aſſume the ſubject at ſome future opportunity, and proceed immediately to the ſecond part.

PART II.

PROP.

To find the ſum of the infinite ſeries $\frac{p}{n \,.\, \overline{n+m} \ldots\ldots \overline{n+rm}} + \frac{q}{\overline{n+m} \ldots\ldots \overline{n+r+1} \,.\, m} + \frac{s}{\overline{n+2m} \ldots\ldots \overline{n+r+2} \,.\, m} +$ &c. *when the laſt differences of the numerators become equal to nothing.*

Aſſume $a + nb + n \,.\, \overline{n+m} \,.\, c + n \,.\, \overline{n+m} \,.\, \overline{n+2m} \,.\, d +$ &c. to any number (r') of terms; then, if for n we write $n+m$, $n+2m$, $n+3m$, &c. ſucceſſively, there will reſult a ſeries of quantities whoſe $\acute{r}$th difference is $=0$; ſubſtitute, therefore, this ſeries of quantities for $p, q, s,$ &c. reſpectively, and the given ſeries becomes

$$\frac{a+nb+n\cdot\overline{n+m}\cdot c+\&c.}{n\cdot\overline{n+m}\ldots\overline{n+rm}}+\frac{a+\overline{n+m}\cdot b+\overline{n+m}\cdot\overline{n+2m}\cdot c+\&c.}{\overline{n+m}\ldots\ldots\overline{n+\overline{r+1}\cdot m}}+$$

$$\frac{a+\overline{n+2m}\cdot b+\overline{n+2m}\cdot\overline{n+3m}\cdot c+\&c.}{\overline{n+2m}\ldots\overline{n+\overline{r+2}\cdot m}}+\&c.$$

which manifeſtly reſolves itſelf into the following ſeries

$$\frac{a}{n\cdot\overline{n+m}\ldots\overline{n+rm}}+\frac{a}{\overline{n+m}\ldots\overline{n+\overline{r+1}\cdot m}}+\frac{a}{\overline{n+2m}\ldots\overline{n+\overline{r+2}\cdot m}}+\&c.$$

$$\frac{b}{\overline{n+m}\ldots\overline{n+rm}}+\frac{b}{\overline{n+2m}\ldots\overline{n+\overline{r+1}\cdot m}}+\frac{b}{\overline{n+3m}\ldots\overline{n+\overline{r+2}\cdot m}}+\&c.$$

$$\frac{c}{\overline{n+2m}\ldots\overline{n+rm}}+\frac{c}{\overline{n+3m}\ldots\overline{n+\overline{r+1}\cdot m}}+\frac{c}{\overline{n+4m}\ldots\overline{n+\overline{r+2}\cdot m}}+\&c.$$

$$\&c. \qquad\qquad \&c. \qquad\qquad \&c.$$

where the number of ſeries is $\acute{r}$, the ſum of each of which being taken by a well known rule, the ſum of the given ſeries becomes

$$\frac{a}{n\cdot\overline{n+m}\ldots\overline{n+\overline{r-1}\cdot m}\cdot m\cdot r}+\frac{b}{\overline{n+m}\ldots\overline{n+\overline{r-1}\cdot m}\cdot m\cdot\overline{r-1}}+$$

$$\frac{c}{\overline{n+2m}\ldots\overline{n+\overline{r-1}\cdot m}\cdot m\cdot\overline{r-2}}+\&c.$$

where the law of continuation is manifeſt.

CASE 1. *To find the ſum of the infinite ſeries* $\frac{3}{1\cdot 2\cdot 3\cdot 4}+\frac{6}{2\cdot 3\cdot 4\cdot 5}+\frac{10}{3\cdot 4\cdot 5\cdot 6}+\frac{15}{4\cdot 5\cdot 6\cdot 7}+\&c.$

Here $n=1$, $m=1$, $r=3$, and the third differences become $=0$; therefore $a+b+2c=3$, $a+2b+6c=6$, $a+3b+12c=10$, conſequently $a=1$, $b=1$, $c=\frac{1}{2}$, and therefore the ſum ſought will be $\frac{1}{1\cdot 2\cdot 3\cdot 3}+\frac{1}{2\cdot 3\cdot 2}+\frac{1}{2\cdot 3}=\frac{11}{36}$.

CASE 2. *To find the ſum of the infinite ſeries* $\frac{1 \cdot 2}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{2 \cdot 3}{3 \cdot 5 \cdot 7 \cdot 9} + \frac{3 \cdot 4}{5 \cdot 7 \cdot 9 \cdot 11} +$ &c.

In this caſe $n = 1$, $m = 2$, $r = 3$, and the 3d differences become $= 0$; therefore $a + b + 3c = 2$, $a + 3b + 15c = 6$, $a + 5b + 35c = 12$, conſequently $a = \frac{1}{4}$, $b = \frac{1}{2}$, $c = \frac{1}{4}$, and hence the ſum of the required ſeries becomes $\frac{3}{1 \cdot 3 \cdot 5 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 5 \cdot 2 \cdot 2 \cdot 2} + \frac{1}{5 \cdot 2 \cdot 4} = \frac{1}{24}$.

CASE 3. *To find the ſum of the infinite ſeries* $\frac{1}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{2}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \frac{4}{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \frac{12}{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \frac{31}{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12} +$ &c.

Here $n = 3$, $m = 1$, $r = 5$, and the 4th differences become $= 0$; therefore $a + 3b + 12c + 60d = 1$, $a + 4b + 20c + 120d = 2$, $a + 5b + 30c + 210d = 4$, $a + 6b + 42c + 336d = 12$, conſequently $a = -46$, $b = 47$, $c = -12$, $d = \frac{5}{6}$, therefore the ſum of the given ſeries is $\frac{-46}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 5} + \frac{47}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 4} - \frac{12}{5 \cdot 6 \cdot 7 \cdot 3} + \frac{5}{6 \cdot 7 \cdot 2 \cdot 6} = \frac{61}{50400}$.

CASE 4. *To find the ſum of the infinite ſeries* $\frac{1}{1 \cdot 4 \cdot 7 \cdot 10} + \frac{9}{4 \cdot 7 \cdot 10 \cdot 13} + \frac{25}{7 \cdot 10 \cdot 13 \cdot 16} +$ &c.

Here $n = 1$, $m = 3$, $r = 3$, and the 3d differences are $= 0$; therefore $a + b + 4c = 1$, $a + 4b + 28c = 9$, $a + 7b + 70c = 25$, conſequently $a = \frac{1}{9}$, $b = \frac{-8}{9}$, $c = \frac{4}{9}$; $\therefore$ the ſum of the given ſeries will be $\frac{1}{1 \cdot 4 \cdot 7 \cdot 3 \cdot 3 \cdot 9} - \frac{8}{4 \cdot 7 \cdot 3 \cdot 2 \cdot 9} + \frac{4}{7 \cdot 3 \cdot 9} = \frac{37}{2268}$.

CASE 5. *To find the Sum of the infinite Series* $\frac{1}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9} + \frac{4}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} + \frac{10}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13} + \frac{20}{7 \cdot 9 \cdot 11 \cdot 13\ 15}$.

In this caſe $n = 1$, $m = 2$, $r = 4$, and the 4th differences become $= 0$; therefore $a + b + 3c + 15d = 1$, $a + 3b + 15c + 105d = 4$, $a + 5b + 35c + 315d = 10$, $a + 7b + 63c + 693d = 20$, conſequently $a = \frac{5}{16}$, $b = \frac{3}{16}$, $c = \frac{1}{16}$, $d = \frac{1}{48}$, and hence the ſum of the given ſeries becomes $\frac{5}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 2 \cdot 4 \cdot 16} + \frac{3}{3 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 16} + \frac{1}{5 \cdot 7 \cdot 2 \cdot 2 \cdot 16} + \frac{1}{7 \cdot 2 \cdot 48} = \frac{1}{384}$.

This propoſition may alſo be applied to find the ſum of all thoſe ſeries whoſe numerators being unity, the denominators ſhall be deficient by any number of correſponding terms, however taken: for as the product of all ſuch factors muſt form a progreſſion, whoſe differences will become equal to nothing, if ſuch products be aſſumed for the numerators of the given ſeries having its factors compleated, another ſeries will be formed equal to the given ſeries, whoſe ſum can be found by this propoſition.

CASE 1. *To find the ſum of the infinite ſeries* $\frac{1}{1 \cdot 2 \cdot 4 \cdot 6} + \frac{1}{2 \cdot 3 \cdot 5 \cdot 7} + \frac{1}{3 \cdot 4 \cdot 6 \cdot 8} +$ &c.

By completing the factors in the denominators, and multiplying the numerators by the ſame quantities the given ſeries becomes $\frac{15}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{24}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{35}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} +$ &c. in which caſe $n = 1$, $m = 1$, $r = 5$, and the 3d differences become

$=0$; therefore $a+b+2c=15$, $a+2b+6c=24$, $a+3b+12c=35$, conſequently $a=8$, $b=5$, $c=1$, and therefore the ſum of the ſeries required is $\frac{8}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 5}+\frac{5}{2\cdot 3\cdot 4\cdot 5\cdot 4}+\frac{1}{3\cdot 4\cdot 5\cdot 3}=\frac{211}{7200}$.

CASE 2. *To find the ſum of the infinite ſeries* $\frac{1}{1\cdot 5\cdot 11}+\frac{1}{3\cdot 7\cdot 13}+\frac{1}{5\cdot 9\cdot 15}+$&c.

This ſeries, when completed, becomes $\frac{189}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}+\frac{495}{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}+\frac{1001}{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}+\frac{1755}{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}+$ &c. where $n=1$, $m=2$, $r=5$, and the 4th differences are $=0$; therefore $a+b+3c+15d=189$, $a+3b+15c+105d=495$, $a+5b+35c+315d=1001$, $a+7b+63c+693d=1755$, conſequently $a=96$, $b=48$, $c=10$, $d=1$; and hence the ſum of the given ſeries is $\frac{96}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 2\cdot 5}+\frac{48}{3\cdot 5\cdot 7\cdot 9\cdot 2\cdot 4}+\frac{10}{5\cdot 7\cdot 9\cdot 2\cdot 3}+\frac{1}{7\cdot 9\cdot 2\cdot 2}=\frac{487}{18900}$.

CASE 3. *To find the ſum of the infinite ſeries* $\frac{1}{1\cdot 4\cdot 13\cdot 16}+\frac{1}{4\cdot 7\cdot 16\cdot 19}+\frac{1}{7\cdot 10\cdot 19\cdot 22}+$&c.

This ſeries reſolves itſelf into $\frac{70}{1\cdot 4\cdot 7\cdot 10\cdot 13\cdot 16}+\frac{130}{4\cdot 7\cdot 10\cdot 13\cdot 16\cdot 19}+\frac{208}{7\cdot 10\cdot 13\cdot 16\cdot 19\cdot 22}+$ &c. where $n=1$, $m=3$, $r=5$, and the 3d differences $=0$; therefore $a+b+4c=70$, $a+4b+28c=130$, $a+7b+70c=208$, from whence $a=54$, $b=12$, $c=1$; therefore the ſum of the given ſeries is

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$$\frac{54}{1 \cdot 4 \cdot 7 \cdot 10 \cdot 13 \cdot 3 \cdot 5} + \frac{12}{4 \cdot 7 \cdot 10 \cdot 13 \cdot 3 \cdot 4} + \frac{1}{7 \cdot 10 \cdot 13 \cdot 3 \cdot 3} = \frac{227}{103800}.$$

By this propoſition we may alſo inveſtigate the ſum of the ſeries when there are any number of deficient terms in the denominators, and where the laſt differences of the numerators become equal to nothing; for if the factors in the denominators be completed, and the numerators be multiplied by the ſame quantities, their differences will ſtill become equal to nothing.

CASE 1. *To find the ſum of the infinite ſeries* $\frac{1}{1 \cdot 3 \cdot 4 \cdot 6} + \frac{3}{2 \cdot 4 \cdot 5 \cdot 7} + \frac{6}{3 \cdot 5 \cdot 6 \cdot 8} + \frac{10}{4 \cdot 6 \cdot 7 \cdot 9} + \frac{15}{5 \cdot 7 \cdot 8 \cdot 10} + \&c.$

This ſeries, by completing the factors in the denominators and multiplying the numerators by the ſame quantities, becomes $\frac{10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{54}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{168}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \&c.$ in which caſe $n=1$, $m=1$, $r=5$, and as the 5th differences are $=0$; $\therefore a+b+2c+6d+24e=10$, $a+2b+6c+24d+120e=54$, $a+3b+12c+60d+360e=168$, $a+4b+20c+120d+840e=400$, $a+5b+30c+210d+1680e=810$, from whence $a=0$, $b=0$, $c=-1$, $d=0$, $e=\frac{1}{2}$, conſequently the ſum of the given ſeries is $= -\frac{1}{3 \cdot 4 \cdot 5 \cdot 3} + \frac{1}{5 \cdot 2} = \frac{17}{180}$.

CASE 2. *To find the ſum of the infinite ſeries* $\frac{1}{1 \cdot 3 \cdot 7 \cdot 9} + \frac{5}{3 \cdot 5 \cdot 9 \cdot 11} + \frac{11}{5 \cdot 7 \cdot 11 \cdot 13} + \frac{19}{7 \cdot 9 \cdot 13 \cdot 15} + \&c.$

By proceeding as before this ſeries becomes $\frac{5}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9} +$

$\frac{35}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} + \frac{99}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13} +$ &c. where $n = 1$, $m = 2$, $r = 4$, and the 4th differences $= 0$; therefore $a + b + 3c + 15d = 5$, $a + 3b + 15c + 105d = 35$, $a + 5d + 35c + 315d = 99$, $a + 7b + 63c + 693d = 209$, conſequently $a = -1$, $b = \frac{3}{4}$, $c = \frac{1}{2}$, $d = \frac{1}{4}$, therefore the ſum of the given ſeries $= \frac{-1}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 2 \cdot 4} + \frac{3}{3 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 7 \cdot 2 \cdot 2 \cdot 2} + \frac{1}{7 \cdot 2 \cdot 4} = \frac{3}{140}$.

By a method ſimilar to that made uſe of in this propoſition may any number of factors be taken from the denominators of thoſe ſeries delivered in part the firſt, and alſo from a great variety of others; but as the examples here given muſt be ſufficient to point out the method of proceeding in all other caſes, we will proceed to the third part.

P A R T III.

THE ſum of every converging infinite ſeries, whoſe terms ultimately become equal to nothing, may always be exhibited by the ſum of another ſeries formed by collecting two or more terms of the former ſeries into one. This is not true, however, where the terms of the infinite ſeries continally diverge, or converge to any aſſignable quantity, and are affected with the ſigns +, −, alternately: for inſtance, the ſeries $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} +$ &c. if we collect two terms into one, beginning at the firſt term, will become $-\frac{1}{2 \cdot 3} - \frac{1}{4 \cdot 5} - \frac{1}{6 \cdot 7} +$ &c. If we begin at the ſecond term it becomes $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} +$ &c.; neither of which gives the ſum of the aſſumed ſeries; but in this, and every other caſe of the like nature, a correction will be neceſſary: to determine the value of which, and from whence the neceſſity thereof ariſes, is the ſubject of this third part.

LEMMA A.

If r be any quantity whatever, then will $\frac{1}{2r} = \frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r} +$ &c. *ad infinitum.*

For $\frac{1}{2r} = \frac{1}{r+r} =$ (by common diviſion) $\frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r} +$ &c. ad infinitum.

Cor. 1. Hence $-\frac{1}{2r} = -\frac{1}{r} + \frac{1}{r} - \frac{1}{r} + \frac{1}{r} -$ &c. ad infinitum.

Cor. 2. Hence alſo $\frac{z}{2v} = \frac{z}{v} - \frac{z}{v} + \frac{z}{v} - \frac{z}{v} +$ &c. ad infinitum;

and $-\frac{z}{2v} = -\frac{z}{v} + \frac{z}{v} - \frac{z}{v} + \frac{z}{v} -$ &c. ad infinitum.

PROP. I.

$\frac{n}{rn+m}$

If ~~[illegible]~~ *be the general term of a ſeries formed by writing for n any ſeries of numbers in arithmetic progreſſion, and whoſe ſigns are alternately + and −; then if a ſeries be formed by col-*

lecting two terms into one, beginning at the firſt term, the ſum of the ſeries thence ariſing will be leſs than the ſum of the given ſeries by $\frac{1}{2r}$. *If a ſeries be formed by beginning at the ſecond Term, the ſum thereof will be greater than the ſum of the given ſeries by* $\frac{1}{2r}$.

For let $\frac{n}{rn+m} - \frac{n+a}{\overline{n+a}\,.\,\overline{r+m}}$ be any two ſucceſſive terms of the ſeries, which, if we begin to collect at the firſt term (the firſt term being +) will be two terms to be collected into one, and which will therefore give $\frac{-am}{\overline{rn+m}\times\overline{n+a}\,.\,\overline{r+m}}$ for a general term of the reſulting ſeries. Let us now make n infinite, and then the denominator of this term becomes infinite, and the numerator finite; therefore the terms of this latter ſeries at an infinite diſtance becoming infinitely ſmall, the ſeries will there terminate. Now, by making n infinite in the given ſeries, the two ſucceſſive general terms at an infinite diſtance become $\frac{1}{r} - \frac{1}{r}$; conſequently *this* ſeries is ſtill continued after the *other* terminates; and the terms of ſuch a continuation will be (as they begin with $\frac{n}{rn+m} - \frac{n+a}{\overline{n+a}\,.\,\overline{r+m}}$ by making n infinite) $\frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r}$ + &c. which will alſo be continued ad infin. and whoſe ſum by the lemma is $\frac{1}{2r}$; conſequently the given ſeries exceeds that which is formed by collecting two terms into one, beginning at the firſt, by $\frac{1}{2r}$; hence the ſum of the *latter* ſeries $+\frac{1}{2r}$ will be equal to the ſum of the *former*. If we begin to collect at the ſecond term, then will $-\frac{n}{rn+m}+\frac{n+a}{n+a\,.\,r+m}$ be the two ſucceſſive general terms of the given ſeries to be collected into one; conſequently the continuation of the given ſeries when n becomes infinite will be $-\frac{1}{r}+\frac{1}{r}-\frac{1}{r}+\frac{1}{r}-$ &c. ad infinitum, whoſe ſum, by cor. 1. to the lem. is $-\frac{1}{2r}$; in this caſe, therefore, the ſum of the given ſeries is leſs than the ſum of the ſeries formed by collecting two terms into one, beginning at the ſecond term, by $\frac{1}{2r}$; hence the ſum of the *latter* ſeries $-\frac{1}{2r}$ will be equal to the ſum of the *former*.

Case 1. *Let the given ſeries be* $\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+$&c.

Here $r=1$, $n=1, 2, 3, 4,$ &c. and $m=1$. Now, if we begin to collect at the firſt term, the ſeries reſolves itſelf into $-\frac{1}{2\,.\,3}-\frac{1}{4\,.\,5}-\frac{1}{6\,.\,7}-$ &c. and the correction, to be *added*, being $\frac{1}{2}$, we have $-\frac{1}{2\,.\,3}-\frac{1}{4\,.\,5}-\frac{1}{6\,.\,7}-$&c. $+\frac{1}{2}$ for the ſum of the given ſeries. Now $-\frac{1}{2\,.\,3}-\frac{1}{4\,.\,5}-\frac{1}{6\,.\,7}-$&c. is well known to be equal to $-1+$ hyp. log. of 2; conſequently the ſum of the given ſeries is $=-\frac{1}{2}+$ hyp. log. of 2.

If we begin to collect at the ſecond term, the ſeries becomes $\frac{1}{1\,.\,2}+\frac{1}{3\,.\,4}+\frac{1}{5\,.\,6}+$&c. and the correction, to be *ſubtracted*, being $\frac{1}{2}$, we have $\frac{1}{1\,.\,2}+\frac{1}{3\,.\,4}+\frac{1}{5\,.\,6}+$&c. $-\frac{1}{2}$ for the ſum of the given ſeries; but $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+$&c. is equal to the hyp. log. of 2; therefore the ſum of the given ſeries is $=-\frac{1}{2}+$ hyp. log. of 2, the ſame as before.

CASE 2. *Let the given ſeries be* $\frac{1}{3}-\frac{2}{5}+\frac{3}{7}+\frac{4}{9}+$&c.

Here $r=2$, $n=1, 2, 3, 4,$ &c. $m=1$. Now, if we begin to collect at the ſecond term, the ſeries becomes $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+$&c. and the correction, to be *ſubtracted*, being $\frac{1}{4}$, we have $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+$&c.$-\frac{1}{4}$ for the ſum of the given ſeries; but $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+$&c. is equal to a circular arc (A) of $22^\circ\frac{1}{2}$, whoſe radius is unity; therefore the ſum of the given ſeries $=A-\frac{1}{4}$.

PROP. II.

If $\frac{x+nz}{w+nv}$ *be the general term of a ſeries formed by writing for n any ſeries of numbers in arithmetic progreſſion, and whoſe terms are alternately + and −; then if a ſeries be formed by collecting two terms into one, beginning at the firſt term, the ſum of the ſeries thence ariſing will be leſs than the ſum of the given ſeries by* $\frac{z}{2v}$. *If a ſeries be formed by beginning at the ſecond term, the ſum thereof will be greater than the ſum of the given ſeries by* $\frac{z}{2v}$.

For let $\frac{x+nz}{w+nv} - \frac{x+\overline{n+a}\,.\,z}{w+\overline{n+a}\,.\,v}$ be any two ſucceſſive termsof the given ſeries, which, if we begin to collect at the firſt term, will be the two general terms to be collected into one, and will therefore give $\frac{axv-awz}{\overline{w+nv}\times\overline{w+\overline{n+a}\,.\,v}}$ for a general term of the reſulting ſeries. Let us now make *n* infinite and then this term will vaniſh, and conſequently the reſulting ſeries will terminate at an infinite diſtance. Now, by making *n* infinite in the given ſeries, the two ſucceſſive terms (as they begin with $\frac{x+nz}{w+nv} - \frac{x+\overline{n+a}\,.\,z}{w+\overline{n+a}\,.\,v}$ by making *n* infinite) become $\frac{z}{v} - \frac{z}{v}$; this ſeries, therefore, is ſtill continued after the other terminates; and the terms of ſuch a continuation will be $\frac{z}{v} - \frac{z}{v} + \frac{z}{v} - \frac{z}{v} +$ &c. ad infinitum, and whoſe ſum by cor. 2. to the lem. is $\frac{z}{2v}$; conſequently the given ſeries exceeds that which is formed by collecting two terms into one, beginning at the firſt, by $\frac{z}{2v}$; hence the ſum of the *latter* ſeries $+\frac{z}{2v}$ will be equal to the ſum of the *former*. Now, if we begin to collect at the ſecond term, then will $-\frac{x+nz}{w+nv} + \frac{x+\overline{n+a}\,.\,z}{w+\overline{n+a}\,.\,v}$ be two general terms of the given ſeries to be collected into one; conſequently the continuation of the given ſeries, when *n* becomes infinite, will be $-\frac{z}{v} + \frac{z}{v} - \frac{z}{v} + \frac{z}{v} -$ &c. ad infinitum, whoſe ſum by cor. 2. to the lem. is $-\frac{z}{2v}$; in this caſe, therefore, the ſum of the given ſeries is leſs than the ſum of the ſeries formed by collecting two terms into one, beginning at the ſecond term, by

$\frac{z}{2v}$; hence the ſum of the *latter* ſeries $-\frac{z}{2v}$ will be equal to the ſum of the *former*.

CASE 1. *Let the given ſeries be* $\frac{7}{3}-\frac{11}{5}+\frac{15}{7}-\frac{19}{9}+$&c.

Here $x=3$, $z=2$, $w=1$, $v=1$, $n=2, 4, 6, 8$, &c. Now, if we begin to collect at the firſt term, the ſeries becomes $\frac{2}{3\cdot 5}+\frac{2}{7\cdot 9}+\frac{2}{11\cdot 13}+$ &c. and the correction, to be *added*, being 1, we have $\frac{2}{3\cdot 5}+\frac{2}{7\cdot 9}+\frac{2}{11\cdot 13}+$ &c. $+\,1$ for the ſum of the given ſeries; but if A = a circular arc of 45° whoſe radius is unity, it is well known that $\frac{2}{3\cdot 5}+\frac{2}{7\cdot 9}+\frac{2}{11\cdot 13}+$ &c. $=1-A$; therefore the ſum of the given ſeries is $2-A$.

CASE 2. *Let the given ſeries be* $\frac{16}{1}-\frac{27}{2}+\frac{38}{3}-\frac{49}{4}+$&c.

Here $w=1$, $v=1$, $x=16$, $z=11$, $n=0, 1, 2, 3$, &c. Now, if we begin to collect at the firſt term, the ſeries becomes $\frac{5}{1\cdot 2}+\frac{5}{3\cdot 4}+\frac{5}{5\cdot 6}+$&c. and the correction, to be *added*, being $\frac{11}{2}$, we have $\frac{5}{1\cdot 2}+\frac{5}{3\cdot 4}+\frac{5}{5\cdot 6}+$&c. $+\frac{11}{2}$ for the ſum of the given ſeries; but $\frac{5}{1\cdot 2}+\frac{5}{3\cdot 4}+\frac{5}{6\cdot 7}+$&c. is equal to 5 × hyp. log. of 2, conſequently the ſum of the given ſeries is equal to $\frac{11}{2}+5$ × hyp. log. of 2.

Because $\frac{axv-awx}{\overline{w+nv}\times\overline{w+n+a}\cdot v}$, the general term of the ſeries formed by reducing two terms into one, has its numerator independent of the value of, *n*, it is manifeſt, that the numerators of that ſeries will be all equal. Now, if a ſeries be aſſumed, the numerators of whoſe terms are unity, and in every other reſpect the ſame as the ſeries in this propoſition, that is, if $\frac{1}{w+nv} - \frac{1}{w+\overline{n+a}.v}$ be two ſucceſſive terms of a ſeries, it is manifeſt, that if every two terms of this ſeries be reduced into one, the general term of the reſulting ſeries will be $\frac{-av}{\overline{w+nv}\times\overline{w+\overline{n+a}.v}}$, where the numerator is a conſtant quantity $-av$; conſequently the ſum of the ſeries whoſe general term is $\frac{avx-awz}{\overline{w+nv}\times\overline{w+\overline{n+a}.v}}$ is to the ſum of the ſeries whoſe general term is $\frac{-av}{\overline{w+nv}\times\overline{w+\overline{n+a}.v}}$ as $vx-wz$ to $-v$, or in a given ratio; whenever, therefore, the ſum of the latter ſeries can be found, the ſum of the former can be found, and conſequently, after proper correction, the ſum of the ſeries in this propoſition can be found.

Hence, therefore, in the two caſes given above in whatever arithmetic progreſſion the numerators may proceed, the ſum of the former can always be expreſſed by circular arcs, and the latter by the hyp. log. of 2.

Hence alſo, as it appears from lem. 1. part the firſt, that the ſum of the ſeries $\frac{1}{1} - \frac{1}{r+1} + \frac{1}{2r+1} - \frac{1}{3r+1} +$ &c. can always be expreſſed by circular arcs and logarithms, it is manifeſt, that if the numerators form any arithmetic progreſſion, the ſum of ſuch ſeries may be found by this propoſition, and will always be exhibited by circular arcs and logarithms.

Beſides the ſeries contained in the foregoing propoſitions, a great variety of other ſeries might be produced where a correction is neceſſary, after collecting two terms into one, in order to exhibit the true value of the given ſeries. As the proper correction, however, may always be found from the principles delivered in the above propoſitions, that is, by conſidering what the terms of the given ſeries become at an infinite diſtance, I ſhall only add one or two inſtances more, and conclude what I at preſent intend to offer on this ſubject.

EX. 1. *Let it be required to find the ſum of the infinite ſeries* $\frac{3\cdot4}{1\cdot2}-\frac{4\cdot5}{2\cdot3}+\frac{5\cdot6}{3\cdot4}-\frac{6\cdot7}{4\cdot5}+$ &c.

This, by reſolving two terms into one, becomes $\frac{16}{1\cdot2\cdot3}+\frac{24}{3\cdot4\cdot5}+\frac{32}{5\cdot6\cdot7}-$ &c.; and as the terms of the given ſeries continually approach to unity, the correction, to be *added*, is $\frac{1}{2}$, conſequently $\frac{16}{1\cdot2\cdot3}+\frac{24}{3\cdot4\cdot5}+\frac{32}{5\cdot6\cdot7}-$ &c. $+\frac{1}{2}$ is equal to the ſum of the given ſeries; but by prop. 1. part I. the ſum of the ſeries $\frac{16}{1\cdot2\cdot3}+\frac{24}{3\cdot4\cdot5}+\frac{32}{5\cdot6\cdot7}+$ &c. is equal to $8S-2$ (S being the hyp. log. 2.) conſequently the ſum of the given ſeries is $8S-1\frac{1}{2}$.

EX. 2. *Let it be required to find the ſum of the infinite ſeries* $\frac{1\cdot2}{1\cdot3}-\frac{2\cdot3}{3\cdot5}+\frac{3\cdot4}{5\cdot7}-\frac{4\cdot5}{7\cdot9}+$ &c.

This ſeries, by reſolving two terms into one, becomes $\frac{4}{1\cdot3\cdot5}+\frac{8}{5\cdot7\cdot9}+\frac{12}{9\cdot11\cdot13}+$ &c. and as the terms of the given ſeries continually approach to $\frac{1}{4}$, the correction, to be *added*, will be $\frac{1}{8}$, therefore $\frac{4}{1.3.5}+\frac{8}{5.7.9}+\frac{12}{9.11.13}+\&c.+\frac{1}{8}$ is = to the ſum of the given ſeries; but by prop. 1. part I. the ſum of $\frac{4}{1.3.5}+\frac{8}{5.7.9}+\frac{12}{9.11.13}+\&c.$ is equal to $\frac{1}{4}S+\frac{1}{8}$ (S being a circular arc of 45°, whoſe radius is unity) hence the ſum of the given ſeries is $\frac{1}{4}S+\frac{1}{4}$.

This method is not only applicable to thoſe caſes, where the given ſeries reſolves itſelf into another, whoſe ſum is either accurately known or can be expreſſed by circular arcs and logarithms, but alſo to thoſe caſes where we want to approximate to the value of the given ſeries, as it muſt, in general, be neceſſary firſt to render the terms of the ſeries converging, by collecting two into one, before the operation of approximation begins, and conſequently a correction of this latter is neceſſary in order to exhibit the value of the given ſeries.

ERRATUM.

Page 20. l. 3. from the bottom, *for* $\frac{rn+m}{n}$ *read* $\frac{n}{rn+m}$.